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(6) STABILITY OF A THICK ELASTIC PLATE UNDER THRUST,

by

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Stability of a Thick Elastic Plate Under Thrust

by

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ABSTRACT

When a rectangular plate of incompressible neo-Hookean elastic material is subjected to a thrust, bifurcations of the flexural or barreling types become possible at certain critical values of the compression ratio. The states of pure homogeneous deformation corresponding to these critical compression ratios are states of neutral equilibrium. Their stability is investigated on the basis of an energy criterion, without restriction on the thickness of the plate.

The critical state corresponding to the lowest order flexural mode is found to be stable (unstable) if the aspect ratio (thickness/length) is less (greater) than about 0.2. Agreement with the classical Euler theory is established in the limiting case of small aspect ratio.

1. Introduction

We consider a rectangular plate of incompressible isotropic elastic material, with neo-Hookean strain-energy function, to be situated with its edges parallel to the axes of a rectangular cartesian coordinate system x . The plate is acted on by forces, applied as dead-loads normally to the faces of the plate which are perpendicular to the 1 and 3 axes of the reference system. The faces perpendicular to the 2-axis are force-free. The constraints on the 1 and 3-faces are such as to permit the plate to undergo pure homogeneous deformations under the action of the applied forces. The load in the 1-direction is assumed to be a thrust and at certain critical values of this thrust bifurcations occur.

If the load in the 3-direction is a tension and the 3-dimension of the plate is sufficiently large, we may assume that these bifurcations are plane strains in the 12-plane, possibly superposed on a uniform extension in the 3-direction. These plane strains may be of the flexural or barreling types. The latter have little practical interest (see §8). They are discussed in the present paper in the interest of completeness, since their inclusion involves negligible complication of the discussion.

The critical compression ratios in the 1-direction at which bifurcations occur have been previously discussed by a number of authors with various degrees of generality. For flexural bifurcations of a neo-Hookean incompressible material they were first determined by Biot [1]. They were determined by Sawyers and Rivlin [2], for an arbitrary incompressible isotropic elastic material, in both the flexural and barreling cases.

In the present paper we discuss the stability of the states of pure homogeneous deformation at which bifurcations occur. The stability criterion employed is the energy criterion and the procedure used is essentially that of Koiter [3,4,5]*. An equilibrium state of the system is a state of deformation at which the total potential energy of the system has a stationary value, i.e. at which its first variation is zero. This state will be stable or unstable accordingly as this stationary value is or is not a proper minimum, i.e. accordingly as the second variation of the potential energy is or is not positive definite.

A state of pure homogeneous strain corresponding to a bifurcation is one of <u>neutral</u> equilibrium. It will be stable if the potential energy is greater for every state lying in some neighborhood of it, which satisfies the kinematic constraints. It will be unstable if the potential energy is less for some such state. (The neighborhood is limited to sufficiently small deformations in the 12-plane superposed on a uniform extension in the 3-direction.) We determine whether this is the case by calculating the stationary value of this excess potential energy. The critical pure homogeneous deformation is stable (unstable) if this stationary value is positive (negative).

These calculations, which are rather cumbersome, are carried out without restriction on the magnitude of the aspect ratio (2-dimension/1-dimension) of the plate. They lead to the conclusion that, for the lowest mode of flexural bifurcation, the pure homogeneous state is stable, provided that the aspect ratio is less than about 0.2 and is unstable for higher aspect ratios. The stability

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* In formulating the problem we have had the benefit of extensive discussion and correspondence with Professor Koiter, for which we are extremely grateful. Most of his suggestions have been incorporated in this paper, which owes much to his generous help and advice.

at low aspect ratios is studied by means of an asymptotic formula for the stationary value of the excess potential energy which is valid up to degree 5 in the aspect ratio. This agrees well with the exact calculations up to fairly high aspect ratios. Moreover, in the limiting case as the aspect ratio tends to zero, it agrees precisely with the result derived by Euler on the basis of elastica theory.

2. Statement of the problem

We consider a rectangular plate of incompressible neo-Hookean elastic material, which has its edges parallel to the axes of a rectangular cartesian coordinate system x . Let $\underset{\sim}{\xi}$ be the vector position, relative to the origin of the system x , of a generic particle of the plate in its undeformed state (state 0) and let its bounding surfaces in this state be

$$\xi_1 = \pm\ell_1 , \quad \xi_2 = \pm\ell_2 , \quad \xi_3 = \pm\ell_3 \quad (\ell_3 >> \ell_1, \ell_2) . \qquad (2.1)$$

We suppose that the plate is maintained in an equilibrium state of pure homogeneous deformation (state I), with extension ratios $\lambda_1, \lambda_2, \lambda_3$ and principal directions parallel to the coordinate axes, by normal forces applied to the surfaces $\xi_1 = \pm\ell_1$ and $\xi_3 = \pm\ell_3$, the surfaces $\xi_2 = \pm\ell_2$ remaining force-free. Let the resultant loads applied to the surfaces $\xi_1 = \pm\ell_1$ and $\xi_3 = \pm\ell_3$ be $\pm R_1$ and $\pm R_3$ respectively and suppose that $R_1 \leq 0$ (thrust).

In state I, we suppose that the surfaces initially at $\xi_1 = \pm\ell_1$ and $\xi_3 = \pm\ell_3$ are constrained so that, in the deformation, they move parallel to the 1 and 3-axes respectively, but points on them are free to move normal to these directions (i.e. the tangential components of the surface tractions are zero).

Let $\underset{\sim}{X}$ be the vector position in state I of the particle which has vector position $\underset{\sim}{\xi}$ in state 0 . Then,

$$X_1 = \lambda_1\xi_1 , \quad X_2 = \lambda_2\xi_2 , \quad X_3 = \lambda_3\xi_3 , \quad \lambda_1\lambda_2\lambda_3 = 1. \quad (2.2)$$

We now consider that the plate undergoes a further deformation which is the superposition of a uniform stretch in the 3-direction and a plane deformation parallel to the 12-plane. We call the state which is then reached state II. We suppose that the particle which is at $\underset{\sim}{X}$ in state I moves to $\underset{\sim}{x}$ in state II, where

$$\underset{\sim}{x} = \underset{\sim}{X} + \underset{\sim}{u} , \tag{2.3}$$

and

$$u_1 = u_1(\xi_1,\xi_2),\ u_2 = u_2(\xi_1,\xi_2),\ u_3 = \lambda_3 E\xi_3 . \tag{2.4}$$

Since the material is incompressible $\underset{\sim}{u}$ must satisfy the relation

$$(1+E)(\lambda_1 u_{2,2}+\lambda_2 u_{1,1} + u_{1,1}u_{2,2} - u_{1,2}u_{2,1}) + \lambda_1\lambda_2 E = 0 . \tag{2.5}$$

Let W_I and W_{II} denote the strain energies, per unit volume, in states I and II respectively. Since the material of the plate is neo-Hookean, they are given, in appropriate units, by

$$\begin{aligned} W_I &= \tfrac{1}{2}(\lambda_1^2+\lambda_2^2+\lambda_3^2-3) , \\ W_{II} &= \tfrac{1}{2}[x_{1,1}^{\ 2} + x_{2,2}^{\ 2} + x_{1,2}^{\ 2} + x_{2,1}^{\ 2} + \lambda_3^2(1+E)^2 -3] . \end{aligned} \tag{2.6}$$

With (2.2), (2.3) and (2.4), we obtain from (2.6)

$$\begin{aligned} W_{II} - W_I = {}& \lambda_1 u_{1,1} + \lambda_2 u_{2,2} + \lambda_3^2 E \\ & + \tfrac{1}{2}(u_{1,1}^{\ 2} + u_{2,2}^{\ 2} + u_{1,2}^{\ 2} + u_{2,1}^{\ 2} + \lambda_3^2 E^2) . \end{aligned} \tag{2.7}$$

We consider that state I is an equilibrium state. We consider also that in state II the resultant loads on the faces of the plate are the same as in state I and that, while state II is not necessarily an equilibrium state, in it the particles of the plate are at rest, at any rate instantaneously.

The Piola-Kirchhoff stress in state I, denoted $\Pi_{\alpha i}$ is given by

$$\Pi_{11} = \lambda_1 - \frac{\lambda_2^2}{\lambda_1}, \quad \Pi_{33} = \lambda_3 - \frac{\lambda_2^2}{\lambda_3}, \tag{2.8}$$

the remaining components of $\Pi_{\alpha i}$ being zero. Then,

$$R_1 = 4\ell_2\ell_3\Pi_{11}, \quad R_3 = 4\ell_1\ell_2\Pi_{33}. \tag{2.9}$$

It follows that, in passing from state I to state II, the increase in potential energy of the loads is

$$- R_1 u_1 \Big|_{\xi_1=-\ell_1}^{\ell_1} - 2R_3\lambda_3 E\ell_3$$

$$= - 2\ell_3 \int_{-\ell_2}^{\ell_2} \int_{-\ell_1}^{\ell_1} (\Pi_{11}u_{1,1} + \Pi_{33}\lambda_3 E) d\xi_1 d\xi_2 . \tag{2.10}$$

Let G_I and G_{II} be the potential energies of the system (plate and load) in states I and II respectively. From (2.7) and (2.10), we obtain, with (2.8),

$$G \overset{\text{def.}}{=} G_{II} - G_I = 2\ell_3 \int_{-\ell_2}^{\ell_2} \int_{-\ell_1}^{\ell_1} \left\{ \frac{\lambda_2^2}{\lambda_1} u_{1,1} + \lambda_2 u_{2,2} + \lambda_2^2 E \right.$$

$$\left. + \frac{1}{2}({u_{1,1}}^2 + {u_{2,2}}^2 + {u_{1,2}}^2 + {u_{2,1}}^2 + \lambda_3^2 E^2) \right\} d\xi_1 d\xi_2 \ . \qquad (2.11)$$

Using (2.5) to substitute for $u_{2,2}$ in (2.11) we can rewrite (2.11) as

$$G = 2\ell_3 \int_{-\ell_2}^{\ell_2} \int_{-\ell_1}^{\ell_1} \left\{ \frac{\lambda_2}{\lambda_1} (u_{1,2}u_{2,1} - u_{1,1}u_{2,2}) + \frac{\lambda_2^2 E^2}{1+E} \right.$$

$$\left. + \frac{1}{2}({u_{1,1}}^2 + {u_{2,2}}^2 + {u_{1,2}}^2 + {u_{2,1}}^2 + \lambda_3^2 E^2) \right\} d\xi_1 d\xi_2 \ . \qquad (2.12)$$

The necessary and sufficient condition for stability of the plate in state I, under the specified loading conditions, is that G be positive definite for all $\underset{\sim}{u}$ and E lying in a neighborhood of $\underset{\sim}{u} = 0$, $E = 0$ and satisfying the constraint (2.5) throughout the plate, as well as the kinematic constraints

$$\begin{aligned} &E = \text{constant everywhere}, \\ &u_1 = \text{constant on} \quad \xi_1 = \pm\ell_1 \ . \end{aligned} \qquad (2.13)$$

We shall investigate the validity of this condition when state I is a critical state for existence of a bifurcation solution.

3. The bifurcation solution

A necessary condition for stability of state I is that the second variation of G be non-negative for all sufficiently small, kinematically admissible values of u_1, u_2, E. This second variation $G_2[u_1,u_2,E]$ is given, from (2.12), by

$$G_2[u_1,u_2,E] = 2\ell_3 \int_{-\ell_2}^{\ell_2} \int_{-\ell_1}^{\ell_1} \left\{ \frac{\lambda_2}{\lambda_1} (u_{1,2}u_{2,1} - u_{1,1}u_{2,2}) + \lambda_2^2 E^2 + \frac{1}{2}(u_{1,1}^{\ 2} + u_{2,2}^{\ 2} + u_{1,2}^{\ 2} + u_{2,1}^{\ 2} + \lambda_3^2 E^2) \right\} d\xi_1 d\xi_2 . \qquad (3.1)$$

u_1, u_2, E must satisfy the kinematic constraint implied by the linearized incompressibility condition

$$\frac{1}{\lambda_1} u_{1,1} + \frac{1}{\lambda_2} u_{2,2} + E = 0 , \qquad (3.2)$$

which is obtained from (2.5), and by the conditions (2.13).

A sufficient condition for stability of state I is that $G_2[u_1,u_2,E]$ be positive definite. Thus, state I is at the <u>stability limit</u> if $G_2[u_1,u_2,E]$ has a zero minimum for some non-vanishing u_1, u_2, E.

We shall now determine the values of u_1, u_2, E for which G_2 has a stationary value, subject to kinematic constraints (3.2) and (2.13). These are given by setting the first variation δG_2 of G_2 equal to zero. We take account of the constraint (3.2) by the method of Lagrange multipliers and denote the multiplier of (3.2) by $-2\ell_3 p$. We then obtain, with the notation $\lambda = \lambda_2/\lambda_1$,

$$\delta G_2 = 2\ell_3 \int_{-\ell_2}^{\ell_2} \int_{-\ell_1}^{\ell_1} \Big\{ (u_{1,1} - \lambda u_{2,2} - \frac{p}{\lambda_1})\delta u_{1,1} + (u_{2,2} - \lambda u_{1,1} - \frac{p}{\lambda_2})\delta u_{2,2}$$

$$+ (u_{1,2} + \lambda u_{2,1})\delta u_{1,2} + (u_{2,1} + \lambda u_{1,2})\delta u_{2,1}$$

$$+ (2\lambda_2^2 E + \lambda_3^2 E - p)\delta E \Big\} d\xi_1 d\xi_2 = 0 . \tag{3.3}$$

This relation may be rewritten as

$$\int_{-\ell_2}^{\ell_2} \int_{-\ell_1}^{\ell_1} \Big\{ (u_{1,11} + u_{1,22} - \frac{p_{,1}}{\lambda_1})\delta u_1 + (u_{2,11} + u_{2,22} - \frac{p_{,2}}{\lambda_2})\delta u_2$$

$$- (2\lambda_2^2 E + \lambda_3^2 E - p)\delta E \Big\} d\xi_1 d\xi_2$$

$$- \int_{-\ell_2}^{\ell_2} [(u_{1,1} - \lambda u_{2,2} - \frac{p}{\lambda_1})\delta u_1 + (u_{2,1} + \lambda u_{1,2})\delta u_2]_{\xi_1 = -\ell_1}^{\ell_1} d\xi_2$$

$$- \int_{-\ell_1}^{\ell_1} [(u_{2,2} - \lambda u_{1,1} - \frac{p}{\lambda_2})\delta u_2 + (u_{1,2} + \lambda u_{2,1})\delta u_1]_{\xi_2 = -\ell_2}^{\ell_2} d\xi_1 = 0 . \tag{3.4}$$

Equation (3.4) yields, with (2.13),

$$\begin{aligned} & u_{1,11} + u_{1,22} - \frac{1}{\lambda_1} p_{,1} = 0 , \\ & u_{2,11} + u_{2,22} - \frac{1}{\lambda_2} p_{,2} = 0 , \\ & 4\ell_1 \ell_2 (2\lambda_2^2 + \lambda_3^2) E = \int_{-\ell_2}^{\ell_2} \int_{-\ell_1}^{\ell_1} p \, d\xi_1 d\xi_2 . \end{aligned} \tag{3.5}$$

Also

$$\left. \begin{aligned} & \lambda u_{2,1} + u_{1,2} = 0 , \\ & u_{2,2} - \lambda u_{1,1} - \frac{p}{\lambda_2} = 0 , \end{aligned} \right\} \quad \text{on } \xi_2 = \pm \ell_2 \tag{3.6}$$

and, with (2.13),

$$
\left.\begin{aligned}
&u_{2,1} = u_{1,2} = 0 , \\
&\int_{-\ell_2}^{\ell_2} (u_{1,1} - \lambda u_{2,2} - \frac{p}{\lambda_1}) d\xi_2 = 0 ,
\end{aligned}\right. \qquad \text{on } \xi_1 = \pm \ell_1 . \tag{3.7}
$$

Equations (3.5)-(3.7), or equivalent equations, have been previously obtained by a number of workers (see, for example, [1,2]) using somewhat different procedures. They yield, with (3.2), solutions of the forms

$$
\begin{aligned}
&u_1 = \left.\begin{matrix} -\sin\Omega\xi_1 \\ \\ \cos\Omega\xi_1 \end{matrix}\right\} \frac{U'}{\lambda\Omega} , \qquad u_2 = \left.\begin{matrix} \cos\Omega\xi_1 \\ \\ \sin\Omega\xi_1 \end{matrix}\right\} U , \\
&p = \left.\begin{matrix} \cos\Omega\xi_1 \\ \\ \sin\Omega\xi_1 \end{matrix}\right\} \frac{\lambda_1 \beta'}{\lambda\Omega^2} , \qquad E = 0 ,
\end{aligned} \tag{3.8}
$$

where

$$
\Omega = n\pi/2\ell_1 \quad (n=1,2,\ldots) , \tag{3.9}
$$

the upper (lower) solution corresponding to n even (odd). In (3.8) U is a function of ξ_2 only and the prime denotes differentiation with respect to ξ_2. β is defined by

$$
\beta = U'' - \Omega^2 U , \tag{3.10}
$$

and U satisfies the differential equation

$$U^{(iv)} - (\lambda^2+1)\Omega^2 U'' + \lambda^2\Omega^4 U = 0 \tag{3.11}$$

and the boundary conditions

$$\left.\begin{aligned} &U'' + \lambda^2\Omega^2 U = 0 \ , \\ &U''' - (2\lambda^2+1)\Omega^2 U' = 0 \ , \end{aligned}\right\} \quad \text{on } \xi_2 = \pm\ell_2 \ . \tag{3.12}$$

We obtain two possible solutions to (3.11) and (3.12). With the notation

$$\eta = \Omega\ell_2 \ , \tag{3.13}$$

one of these solutions may be written as

$$\begin{aligned} U(\xi_2) &= M(\cosh\lambda\Omega\xi_2 - m\cosh\Omega\xi_2) \ , \\ m &= \frac{2\lambda^2\cosh\lambda\eta}{(\lambda^2+1)\cosh\eta} \ , \end{aligned} \tag{3.14}$$

where λ and η satisfy the secular equation

$$4\lambda^3\tanh\eta = (\lambda^2+1)^2\tanh\lambda\eta \ . \tag{3.15}$$

The other solution may be written as

$$\begin{aligned} U(\xi_2) &= M(\sinh\lambda\Omega\xi_2 - m\sinh\Omega\xi_2) \ , \\ m &= \frac{2\lambda^2\sinh\lambda\eta}{(\lambda^2+1)\sinh\eta} \ , \end{aligned} \tag{3.16}$$

where λ and η satisfy the secular equation

$$4\lambda^3 \tanh\lambda\eta = (\lambda^2+1)^2 \tanh\eta \ . \tag{3.17}$$

We note that for any specified η , each of the equations (3.15) and (3.17) yields a unique value for λ .

The deformations described by (3.14) are antisymmetric with respect to the 13-plane, i.e. they are flexural deformations. Those described by (3.16) are symmetric with respect to the 13-plane, i.e. they are barreling deformations.

The two secular equations (3.15) and (3.17) can be rewritten as

$$\frac{\sinh(\lambda+1)\eta}{\sinh(\lambda-1)\eta} = \nu \ \frac{(\lambda+1)\{\lambda(\lambda+1)^2+(\lambda-1)^2\}}{(\lambda-1)\{(\lambda+1)^2-\lambda(\lambda-1)^2\}} \ , \tag{3.18}$$

where $\nu = 1$ for flexural deformations (equation (3.15)) and $\nu = -1$ for barreling deformations (equation (3.17)).

We note that each of the pairs $(3.14)_2$, (3.15) and $(3.16)_2$, (3.17) leads to the result

$$m^2 = \frac{\lambda \sinh 2\lambda\eta}{\sinh 2\eta} \ . \tag{3.19}$$

By substituting in (3.1) either the upper or lower expressions (3.8), we obtain, by using (3.11), the result

$$G_2 = \frac{\ell_3 \ell_1}{\lambda^2 \Omega^2} \int_{-\ell_2}^{\ell_2} \left\{ (3\lambda^2+1)\Omega^2 (UU')' - (UU''')' + (U'U'')' \right\} d\xi_2 \ . \tag{3.20}$$

We now carry out the integration and use (3.12) to obtain

$$G_2 = 0 . \tag{3.21}$$

We conclude that the critical state for which G_2 has a stationary value is a state of neutral equilibrium.

4. Stability in the critical case of neutral equilibrium

We have seen in §2 that the necessary and sufficient condition for the plate to be stable in state I is that G, given by (2.12), shall be positive definite for all u_1, u_2, E lying in a neighborhood of $u_1 = u_2 = E = 0$ and satisfying the kinematic constraints (2.5) and (2.13). We now develop this condition further in the case when state I is a critical state of neutral equilibrium discussed in §3.

Denoting by $\hat{u}_1$, $\hat{u}_2$, $\hat{E} = 0$, $\hat{p}$ the values of u_1, u_2, E, p given by equations (3.8), we decompose an arbitrary u_1, u_2, E in the following manner:

$$u_1 = a\hat{u}_1 + v_1 \,, \quad u_2 = a\hat{u}_2 + v_2 \,, \quad E = F \,, \tag{4.1}$$

where v_1, v_2 is orthogonal to $\hat{u}_1$, $\hat{u}_2$, i.e.

$$\int_{-\ell_2}^{\ell_2} \int_{-\ell_1}^{\ell_1} (\hat{u}_1 v_1 + \hat{u}_2 v_2) d\xi_1 d\xi_2 = 0 \,, \tag{4.2}$$

and a is given by

$$a = \frac{\displaystyle\int_{-\ell_2}^{\ell_2} \int_{-\ell_1}^{\ell_1} (\hat{u}_1 u_1 + \hat{u}_2 u_2) d\xi_1 d\xi_2}{\displaystyle\int_{-\ell_2}^{\ell_2} \int_{-\ell_1}^{\ell_1} (\hat{u}_1^2 + \hat{u}_2^2) d\xi_1 d\xi_2} \,. \tag{4.3}$$

Substituting from (4.1) in (2.12) and using (3.1), (3.5)-(3.7) and (3.21), we obtain

$$G = 2\ell_3 \int_{-\ell_2}^{\ell_2} \int_{-\ell_1}^{\ell_1} \left\{ \frac{1}{2}(v_{1,1}^{\ 2}+v_{2,2}^{\ 2}+v_{1,2}^{\ 2}+v_{2,1}^{\ 2}+\lambda_3^2 F^2) + \lambda(v_{1,2}v_{2,1}-v_{1,1}v_{2,2}) + \frac{\lambda_2^2 F^2}{1+F} + a\hat{p}(\frac{1}{\lambda_1} v_{1,1}+ \frac{1}{\lambda_2} v_{2,2}) \right\} d\xi_1 d\xi_2 , \tag{4.4}$$

where $\hat{p}$ is given by (3.8). From the incompressibility condition (2.5) and (3.2), it follows that v_1, v_2, F must satisfy the relation

$$\lambda_1 v_{2,2} + \lambda_2 v_{1,1} + \frac{\lambda_1\lambda_2 F}{1+F} + a^2(\hat{u}_{1,1}\hat{u}_{2,2}-\hat{u}_{1,2}\hat{u}_{2,1}) + a(\hat{u}_{1,1}v_{2,2} + \hat{u}_{2,2}v_{1,1} - \hat{u}_{1,2}v_{2,1} - \hat{u}_{2,1}v_{1,2}) + v_{1,1}v_{2,2} - v_{1,2}v_{2,1} = 0 . \tag{4.5}$$

From (3.8) we obtain, by carrying out the integration with respect to ξ_1 ,

$$\int_{-\ell_2}^{\ell_2} \int_{-\ell_1}^{\ell_1} \hat{p} d\xi_1 d\xi_2 = 0 , \quad \int_{-\ell_2}^{\ell_2} \int_{-\ell_1}^{\ell_1} \hat{p}(\hat{u}_{1,1}\hat{u}_{2,2}-\hat{u}_{1,2}\hat{u}_{2,1}) d\xi_1 d\xi_2 = 0 . \tag{4.6}$$

Then, using (4.5) to eliminate the term in (4.4) which is linear in $v_{1,1}$ and $v_{2,2}$, we obtain with (4.6)

$$G = 2\ell_3 \int_{-\ell_2}^{\ell_2} \int_{-\ell_1}^{\ell_1} \Big\{ \tfrac{1}{2}(v_{1,1}^{\;2}+v_{2,2}^{\;2}+v_{1,2}^{\;2}+v_{2,1}^{\;2}+\lambda_3^2F^2)$$

$$+ \lambda(v_{1,2}v_{2,1}-v_{1,1}v_{2,2}) + \frac{\lambda_2^2F^2}{1+F}$$

$$- a^2\lambda_3\hat{p}(\hat{u}_{1,1}v_{2,2}+\hat{u}_{2,2}v_{1,1}-\hat{u}_{1,2}v_{2,1}-\hat{u}_{2,1}v_{1,2})$$

$$- a\lambda_3\hat{p}(v_{1,1}v_{2,2}-v_{1,2}v_{2,1}) \Big\} d\xi_1 d\xi_2 \ . \tag{4.7}$$

It follows from (2.13), (4.1) and (3.7) that

$$v_{1,2} = 0 \quad \text{for} \quad \xi_1 = \pm\ell_1 \ . \tag{4.8}$$

For a fixed small value of a, we now determine the values of v_1, v_2, F, satisfying *the kinematic constraints* (4.5) and (4.8) and the orthogonality condition (4.2), for which G has a stationary value. Since these values must be $O(a^2)$, we introduce the notation

$$v_1 = a^2\bar{u}_1 \ , \quad v_2 = a^2\bar{u}_2 \ , \quad F = a^2\bar{E} \ . \tag{4.9}$$

Then, neglecting terms of higher degree than the fourth in a, we obtain from (4.7),

$$G = a^4\bar{G} = a^4 2\ell_3 \int_{-\ell_2}^{\ell_2} \int_{-\ell_1}^{\ell_1} \Big\{ \tfrac{1}{2}(\bar{u}_{1,1}^{\;2}+\bar{u}_{2,2}^{\;2}+\bar{u}_{1,2}^{\;2}+\bar{u}_{2,1}^{\;2}+\lambda_3^2\bar{E}^2)$$

$$+ \lambda(\bar{u}_{1,2}\bar{u}_{2,1}-\bar{u}_{1,1}\bar{u}_{2,2}) + \lambda_2^2\bar{E}^2$$

$$- \lambda_3\hat{p}(\hat{u}_{1,1}\bar{u}_{2,2}+\hat{u}_{2,2}\bar{u}_{1,1}-\hat{u}_{1,2}\bar{u}_{2,1}-\hat{u}_{2,1}\bar{u}_{1,2}) \Big\} d\xi_1 d\xi_2 \ . \tag{4.10}$$

Neglecting terms of higher degree than the second in a^2, the incompressibility condition (4.5) can be written as

$$a^2[\frac{1}{\lambda_1}\bar{u}_{1,1}+\frac{1}{\lambda_2}\bar{u}_{2,2}+\bar{E}+\lambda_3(\hat{u}_{1,1}\hat{u}_{2,2}-\hat{u}_{1,2}\hat{u}_{2,1})] = 0 \qquad (4.11)$$

and the orthogonality constraint (4.2) may be written as

$$a^2\int_{-\ell_2}^{\ell_2}\int_{-\ell_1}^{\ell_1}(\hat{u}_1\bar{u}_1+\hat{u}_2\bar{u}_2)d\xi_1 d\xi_2 = 0 \ . \qquad (4.12)$$

We determine the values of $\bar{u}_1$, $\bar{u}_2$, $\bar{E}$ for which $\bar{G}$ has a stationary value by equating its first variation to zero, relaxing the constraints (4.11) and (4.12) by the method of Lagrange multipliers. We denote the Lagrange multipliers for the constraints (4.11) and (4.12) by $-2\ell_3 a^2\bar{p}$ and $2\ell_3 a^2\chi$ respectively, where $\bar{p}$ is a function of ξ_1,ξ_2 and χ is a constant. We thus obtain from (4.10), after some algebraic manipulation,

$$\begin{aligned}\delta\bar{G} = 2\ell_3\Bigg\{&-\int_{-\ell_2}^{\ell_2}\int_{-\ell_1}^{\ell_1}(\Phi_1\delta\bar{u}_1+\Phi_2\delta\bar{u}_2-\Phi\delta\bar{E})d\xi_1 d\xi_2\\ &+\int_{-\ell_1}^{\ell_1}\Big[\Theta_1\delta\bar{u}_1+\Theta_2\delta\bar{u}_2\Big]_{\xi_2=-\ell_2}^{\ell_2}d\xi_1\\ &+\int_{-\ell_2}^{\ell_2}\Big[\Psi_1\delta\bar{u}_1+\Psi_2\delta\bar{u}_2\Big]_{\xi_1=-\ell_1}^{\ell_1}d\xi_2\Bigg\} = 0 \ ,\end{aligned} \qquad (4.13)$$

where $\delta\bar{G}$, $\delta\bar{u}_1$, $\delta\bar{u}_2$, $\delta\bar{E}$ denote the variations of $\bar{G}$, $\bar{u}_1$, $\bar{u}_2$, $\bar{E}$ and

$$
\begin{aligned}
\Phi_1 &= \bar{u}_{1,11} + \bar{u}_{1,22} - \frac{1}{\lambda_1}\bar{p}_{,1} - \lambda_3(\hat{p}_{,1}\hat{u}_{2,2} - \hat{p}_{,2}\hat{u}_{2,1}) - \chi\hat{u}_1 , \\
\Phi_2 &= \bar{u}_{2,11} + \bar{u}_{2,22} - \frac{1}{\lambda_2}\bar{p}_{,2} - \lambda_3(\hat{p}_{,2}\hat{u}_{1,1} - \hat{p}_{,1}\hat{u}_{1,2}) - \chi\hat{u}_2 , \\
\Phi &= -\bar{p} + (\lambda_3^2 + 2\lambda_2^2)\bar{E} , \\
\Theta_1 &= \bar{u}_{1,2} + \lambda\bar{u}_{2,1} + \lambda_3\hat{p}\hat{u}_{2,1} , \\
\Theta_2 &= \bar{u}_{2,2} - \lambda\bar{u}_{1,1} - \frac{1}{\lambda_2}\bar{p} - \lambda_3\hat{p}\hat{u}_{1,1} , \\
\Psi_1 &= \bar{u}_{1,1} - \lambda\bar{u}_{2,2} - \frac{1}{\lambda_1}\bar{p} - \lambda_3\hat{p}\hat{u}_{2,2} , \\
\Psi_2 &= \bar{u}_{2,1} + \lambda\bar{u}_{1,2} + \lambda_3\hat{p}\hat{u}_{1,2} .
\end{aligned}
\tag{4.14}
$$

From (4.13) we obtain, bearing in mind that $\delta\bar{E}$ is constant throughout the plate,

$$
\Phi_1 = 0 , \quad \Phi_2 = 0 , \quad \int_{-\ell_2}^{\ell_2}\int_{-\ell_1}^{\ell_1} \Phi d\xi_1 d\xi_2 = 0 . \tag{4.15}
$$

We also obtain, with (4.8), (4.9) and (3.7)

$$
\Theta_1 = \Theta_2 = 0 \quad \text{for} \quad \xi_2 = \pm\ell_2 , \tag{4.16}
$$

and

$$
\left.
\begin{aligned}
&\bar{u}_{2,1} = 0 , \ \bar{u}_1 = \pm\lambda_1\bar{e}\ell_1 , \text{ say} , \\
&\int_{-\ell_2}^{\ell_2} \Psi_1 d\xi_2 = 0 ,
\end{aligned}
\right\} \quad \text{for} \quad \xi_1 = \pm\ell_1 , \tag{4.17}
$$

where $\bar{e}$ is a constant.

Equations (4.15)-(4.17), together with the conditions (4.11) and (4.12), yield the values of $\bar{u}_1$, $\bar{u}_2$, $\bar{E}$, subject to the kinematic and orthogonality constraints, for which G has a stationary value. We discuss the solution of these equations in the next section.

5. The deformation for which $\overline{G}$ has a stationary value

Introducing into (4.14) the expressions (3.8) for $\hat{u}_1$, $\hat{u}_2$, $\hat{p}$, we obtain from (4.15) and (4.11)

$$
\begin{aligned}
&\bar{u}_{1,11} + \bar{u}_{1,22} - \frac{1}{\lambda_1}\bar{p}_{,1} = \frac{\omega}{2\lambda_2\lambda\Omega}(\beta''U - \beta'U')\sin 2\Omega\xi_1 + \chi\hat{u}_1 , \\
&\bar{u}_{2,11} + \bar{u}_{2,22} - \frac{1}{\lambda_2}\bar{p}_{,2} = -\frac{1}{2\lambda^2\lambda_2\Omega^2}\{(\beta'U')' \\
&\qquad\qquad + \omega(\beta''U' - \beta'U'')\cos 2\Omega\xi_1\} + \chi\hat{u}_2 , \qquad (5.1)\\
&\int_{-\ell_2}^{\ell_2}\int_{-\ell_1}^{\ell_1} \bar{p}\,d\xi_1 d\xi_2 = 4\ell_1\ell_2(\lambda_3^2 + 2\lambda_2^2)\overline{E} ,
\end{aligned}
$$

and

$$
2\lambda_2(\bar{u}_{2,2} + \lambda\bar{u}_{1,1} + \lambda_2\overline{E}) = (UU')' - \omega\alpha\cos 2\Omega\xi_1 , \qquad (5.2)
$$

where

$$
\alpha = UU'' - U'^2 , \qquad (5.3)
$$

and $\omega = 1$ or -1 accordingly as the upper or lower solution is taken in (3.8).

Similarly, the boundary conditions (4.16) may be written as

$$
\left.
\begin{aligned}
&\bar{u}_{1,2} + \lambda\bar{u}_{2,1} = \frac{\omega U\beta'}{2\lambda\lambda_2\Omega}\sin 2\Omega\xi_1 , \\
&\bar{u}_{2,2} - \lambda\bar{u}_{1,1} - \frac{\bar{p}}{\lambda_2} = -\frac{U'\beta'}{2\lambda^2\lambda_2\Omega^2}(1 + \omega\cos 2\Omega\xi_1) ,
\end{aligned}
\right\} \text{ on } \xi_2 = \pm\ell_2 . \qquad (5.4)
$$

With the boundary conditions (5.4) and (4.17), we obtain the following solution of equations $(5.1)_{1,2}$ and (5.2), for which $\bar{u}_1$, $\bar{u}_2$ is orthogonal to $\hat{u}_1$, $\hat{u}_2$:

$$
\begin{aligned}
\bar{u}_1 &= \bar{U}_1 \sin 2\Omega\xi_1 + \lambda_1 \bar{e}\xi_1 , \\
\bar{u}_2 &= \bar{U} \cos 2\Omega\xi_1 + \bar{V} , \\
\bar{p} &= \bar{P} \cos 2\Omega\xi_1 + \bar{Q} , \\
\chi &= 0 ,
\end{aligned}
\tag{5.5}
$$

where $\bar{V}$ and $\bar{Q}$ are given by

$$
\begin{aligned}
\bar{V} &= \frac{1}{2\lambda_2} UU' - \lambda_2(\bar{e} + \bar{E})\xi_2 , \\
\bar{Q} &= \frac{1}{2}(UU')' + \frac{1}{2\lambda^2\Omega^2} \beta' U' - \lambda_2^2(2\bar{e} + \bar{E})
\end{aligned}
\tag{5.6}
$$

and $\bar{U}_1$, $\bar{U}$ and $\bar{P}$ are functions of ξ_2 only , which satisfy the ordinary differential equations

$$
\begin{aligned}
\bar{U}_1'' - 4\Omega^2\bar{U}_1 + \frac{2\Omega\bar{P}}{\lambda_1} &= \frac{\omega}{2\lambda\lambda_2\Omega} (\beta'' U - \beta' U') , \\
\bar{U}'' - 4\Omega^2\bar{U} - \frac{\bar{P}'}{\lambda_2} &= - \frac{\omega}{2\lambda^2\lambda_2\Omega^2} (\beta'' U' - \beta' U'') , \\
\bar{U}' + 2\lambda\Omega\bar{U}_1 &= - \frac{\omega\alpha}{2\lambda_2} ,
\end{aligned}
\tag{5.7}
$$

and the boundary conditions

$$
\left.
\begin{aligned}
\bar{U}_1' - 2\lambda\Omega\bar{U} &= \frac{\omega}{2\lambda\lambda_2\Omega} \beta' U , \\
\bar{U}' - 2\lambda\Omega\bar{U}_1 - \frac{\bar{P}}{\lambda_2} &= - \frac{\omega}{2\lambda^2\lambda_2\Omega^2} \beta' U' ,
\end{aligned}
\right\} \text{ on } \xi_2 = \pm \ell_2 .
\tag{5.8}
$$

From $(5.7)_{1,3}$ we obtain

$$\bar{U}_1 = -\frac{1}{2\lambda\Omega}\left(\bar{U}' + \frac{\omega\alpha}{2\lambda_2}\right) ,$$
$$\bar{P} = \frac{\lambda_1}{4\lambda\Omega^2}\{\bar{U}''' - 4\Omega^2\bar{U}' + \frac{\omega}{2\lambda_2}[\alpha'' - 4\Omega^2\alpha + 2(\beta''U - \beta'U')]\} . \tag{5.9}$$

Then, from $(5.7)_2$ we obtain with $(5.9)_2$, (5.3), (3.10) and (3.11)

$$\bar{U}^{(iv)} - 4\Omega^2(\lambda^2+1)\bar{U}'' + 16\lambda^2\Omega^4\bar{U} = -\frac{3\omega\Omega^2}{2\lambda_2}(\lambda^2-1)\alpha' . \tag{5.10}$$

With (3.14) and (3.16), equation (5.3) yields

$$\alpha = \nu M^2\Omega^2\{\lambda^2 + m^2 - \frac{1}{2}m[\nu(\lambda-1)^2\cosh(\lambda+1)\Omega\xi_2$$
$$+ (\lambda+1)^2\cosh(\lambda-1)\Omega\xi_2]\} . \tag{5.11}$$

Substitution of (5.9) in (5.8) yields, with (3.11) and (3.12), the boundary conditions in the form

$$\left.\begin{aligned} &\bar{U}'' + 4\lambda^2\Omega^2\bar{U} = -\frac{\omega}{2\lambda_2}\Omega^2(7\lambda^2+1)UU' , \\ &\bar{U}''' - 4(2\lambda^2+1)\Omega^2\bar{U}' = \frac{2\omega}{\lambda_2}(2\lambda^2-1)\Omega^2U'^2 , \end{aligned}\right\} \text{ on } \xi_2 = \pm\ell_2 . \tag{5.12}$$

We note from (3.14) and (3.16) that

$$U'^2 = \frac{1}{2}M^2\Omega^2\{\lambda^2\cosh 2\lambda\Omega\xi_2 + m^2\cosh 2\Omega\xi_2 - \nu(\lambda^2+m^2)$$
$$- 2\lambda m[\cosh(\lambda+1)\Omega\xi_2 - \nu\cosh(\lambda-1)\Omega\xi_2]\} ,$$
$$UU' = \frac{1}{2}M^2\Omega\{\lambda\sinh 2\lambda\Omega\xi_2 + m^2\sinh 2\Omega\xi_2 \tag{5.13}$$
$$- m[(\lambda+1)\sinh(\lambda+1)\Omega\xi_2 + \nu(\lambda-1)\sinh(\lambda-1)\Omega\xi_2]\} .$$

It can easily be verified, using (5.11), that equation (5.10) has a solution of the form

$$\bar{U} = M_1 \sinh 2\lambda\Omega\xi_2 + M_2 \sinh 2\Omega\xi_2 - M_3 \sinh(\lambda+1)\Omega\xi_2 - M_4 \sinh(\lambda-1)\Omega\xi_2 \ , \tag{5.14}$$

where, provided that $\lambda \neq 3$,

$$\begin{aligned} M_3 &= \frac{3\omega\Omega}{4\lambda_2} mM^2 \frac{(\lambda^2-1)(\lambda+1)}{(3\lambda+1)(\lambda+3)} \ , \\ M_4 &= \frac{3\omega\Omega}{4\lambda_2} \nu mM^2 \frac{(\lambda^2-1)(\lambda-1)}{(3\lambda-1)(\lambda-3)} \ . \end{aligned} \tag{5.15}$$

The integration constants M_1 and M_2 are determined from the boundary conditions (5.12), with (5.13), as

$$\begin{aligned} M_1 &= \frac{\omega\Omega M^2}{32\lambda\lambda_2\Delta} \{4\lambda^2 f_1 \cosh 2\eta - (\lambda^2+1) f_2 \sinh 2\eta\} \ , \\ M_2 &= \frac{\omega\Omega M^2}{16\lambda_2\Delta} \{\lambda f_2 \sinh 2\lambda\eta - (\lambda^2+1) f_1 \cosh 2\lambda\eta\} \ , \end{aligned} \tag{5.16}$$

where

$$\Delta = 4\lambda^3 \sinh 2\lambda\eta \cosh 2\eta - (\lambda^2+1)^2 \cosh 2\lambda\eta \sinh 2\eta \ ,$$

$$\begin{aligned} f_1 &= m\{f(\lambda)\sinh(\lambda+1)\eta - \nu f(-\lambda)\sinh(\lambda-1)\eta\} \\ &\quad -(7\lambda^2+1)(\lambda \sinh 2\lambda\eta + m^2 \sinh 2\eta) \ , \\ f_2 &= m\{g(\lambda)\cosh(\lambda+1)\eta + \nu g(-\lambda)\cosh(\lambda-1)\eta\} \\ &\quad - 4(2\lambda^2-1)\{\lambda^2 \cosh 2\lambda\eta + m^2 \cosh 2\eta - \nu(\lambda^2+m^2)\} \ , \end{aligned} \tag{5.17}$$

with $f(\lambda)$ and $g(\lambda)$ defined by

$$f(\lambda) = (\lambda+1)\{\frac{3(\lambda^2-1)(5\lambda^2+2\lambda+1)}{(3\lambda+1)(\lambda+3)} + 7\lambda^2+1\} ,$$

$$g(\lambda) = \frac{3(\lambda^2-1)(\lambda+1)^2}{(3\lambda+1)(\lambda+3)} (7\lambda^2-2\lambda+3) + 8\lambda(2\lambda^2-1) . \tag{5.18}$$

The constants $\overline{E}$ and $\overline{e}$ can be obtained from the relations $(5.1)_3$ and $(4.17)_3$ which express the dead-loading conditions on the faces $\xi_3 = \pm\ell_3$ and $\xi_1 = \pm\ell_1$ respectively. Using $(5.5)_3$ to substitute for $\overline{p}$ in $(5.1)_3$, we obtain with $(5.6)_2$,

$$(3\lambda_2^2 + \lambda_3^2)\overline{E} + 2\lambda_2^2\overline{e} = -\frac{1}{2} H , \tag{5.19}$$

where

$$H = \frac{1}{2\lambda^2\Omega^2\ell_2} \int_{-\ell_2}^{\ell_2} (U''^2 + \Omega^2U'^2)d\xi_2 . \tag{5.20}$$

Substitution from (5.9) and (3.8) in $(4.14)_6$ yields

$$\begin{aligned}\Psi_1 = &-\frac{1}{4\lambda\Omega^2}\left[\overline{U}'' + 4\lambda^2\Omega^2\overline{U} + \frac{\omega}{2\lambda_2}(\alpha' + 2\beta'U)\right]'\cos 2\Omega\xi_1 \\ &+ \frac{1}{\lambda_1}\left[2\lambda_2^2\overline{E} + (3\lambda_2^2+\lambda_1^2)\overline{e} + \frac{1}{\lambda^2\Omega^2}(U''^2 + \Omega^2U'^2)\right] \\ &- \frac{1}{\lambda_1\lambda^2\Omega^2}\left[U'(U'' + \lambda^2\Omega^2U)\right]' .\end{aligned} \tag{5.21}$$

From $(4.17)_3$ and (5.21) we obtain, with (5.12), (5.3), (3.10)-(3.12) and (5.20),

$$2\lambda_2^2\overline{E} + (3\lambda_2^2+\lambda_1^2)\overline{e} = -H . \tag{5.22}$$

Equations (5.19) and (5.22) yield

$$\bar{E} = \frac{\lambda_3 H}{2\kappa}(\lambda^2-1) \ , \qquad \bar{e} = -\frac{\lambda_3 \lambda H}{\kappa}(2\lambda+\lambda_3^3) \ , \tag{5.23}$$

with

$$\kappa = 5\lambda^3 + 3\lambda(\lambda\lambda_3^3+1) + \lambda_3^3 \ . \tag{5.24}$$

With (3.14), (3.16) and (3.19), equation (5.20) yields

$$H = \frac{1}{2} M^2\Omega^2(\lambda^2-1)\left(\nu + \frac{\lambda^2-3}{\lambda^2+1} \ \frac{\sinh 2\lambda\eta}{2\lambda\eta}\right) \ . \tag{5.25}$$

6. Stationary value of $\bar{G}$

We now use $(5.5)_{1,2}$ and (3.8) to substitute for $\bar{u}_1$, $\bar{u}_2$, $\hat{u}_1$, $\hat{u}_2$, $\hat{p}$ in (4.10) and carry out the integration with respect to ξ_1. We obtain

$$\bar{G} = 2\ell_1\ell_3 \int_{-\ell_2}^{\ell_2} (g_1 + g_2)\, d\xi_2 , \tag{6.1}$$

where

$$\begin{aligned} g_1 &= \lambda_1^2\bar{e}^2 + (2\lambda_2^2+\lambda_3^2)\bar{E}^2 - \bar{e}(2\lambda_2\bar{V}' + \frac{1}{\lambda^2\Omega^2}\beta'U') + \bar{V}'^2 \\ &\quad + \frac{1}{\lambda_2\lambda^2\Omega^2}\bar{V}'\beta'U' , \\ g_2 &= \frac{1}{2}\{4\Omega^2\bar{U}_1^2 + \bar{U}_1'^2 + \bar{U}'^2 + 4\Omega^2\bar{U}^2 - 4\lambda\Omega(\bar{U}\bar{U}_1)' \\ &\quad + \frac{\omega\beta'}{\lambda_2\lambda^2\Omega^2}(U'\bar{U}' - 2\lambda\Omega U'\bar{U}_1 + 2U''\bar{U} - \lambda\Omega U\bar{U}_1')\} . \end{aligned} \tag{6.2}$$

By using $(5.6)_1$, (5.19) and (5.22) in $(6.2)_1$ and (5.9), (5.3), (5.10) in $(6.2)_2$, together with (3.10) and (3.11), we obtain

$$\begin{aligned} g_1 &= \frac{2\bar{e}+\bar{E}}{\lambda^2\Omega^2}\{U''^2 + \Omega^2U'^2 - \frac{1}{2}\lambda^2\Omega^2 H - [U'(U''+\lambda^2\Omega^2U)]'\} \\ &\quad + \frac{1}{4\lambda_2^2}(UU')'[(UU')' + \frac{2}{\lambda^2\Omega^2}\beta'U'] , \\ g_2 &= -\frac{3\omega}{16\lambda_2\lambda^2\Omega^2}\{\beta(U''+\lambda^2\Omega^2U) - 2\beta'U'\}\bar{U}' \\ &\quad + \frac{1}{32\lambda_2^2\lambda^2\Omega^2}\{\alpha'^2 + 4\Omega^2\alpha^2 + 4\alpha(\beta'U' - \lambda^2\Omega^2\beta U)\} \\ &\quad + \frac{1}{8\lambda^2\Omega^2}\{\bar{U}'(\bar{U}'' + 4\lambda^2\Omega^2\bar{U}) - \bar{U}[\bar{U}''' - 4\Omega^2(2\lambda^2+1)\bar{U}'] \end{aligned} \tag{6.3}$$

$$+ \frac{\omega}{\lambda_2}\bar{U}'(3UU''' - U'U'' - 2\Omega^2 UU')$$

$$+ \frac{\omega}{2\lambda_2}\bar{U}[6U''^2 + 3\Omega^2(\lambda^2+1)UU'' - \Omega^2 U'^2(5\lambda^2+11)$$

$$+ 3\Omega^2(\lambda^2-1)\alpha + 6\beta'U' + (2U'' - 3\beta)(U'' + \lambda^2\Omega^2 U)]$$

$$+ \frac{1}{\lambda_2^2}\alpha\beta'U\}' \ .$$

In deriving the relation $(6.3)_2$, the following identity is used:

$$\{\Omega^2(\lambda^2-1)\alpha - \beta(U'' + \lambda^2\Omega^2 U) + 2\beta'U'\}' \equiv 0 \ . \qquad (6.4)$$

We now substitute from (6.3) in (6.1) and use the boundary conditions (3.12) and (5.12), together with (5.23), to obtain

$$\bar{G} = - 4\ell_1\ell_2\ell_3(\Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 - \Gamma_5) \ , \qquad (6.5)$$

where

$$\begin{aligned}
\Gamma_1 &= \frac{\lambda_3}{4\kappa}(7\lambda^2 + 4\lambda\lambda_3^3 + 1)H^2 \ , \\
\Gamma_2 &= \frac{1}{4\lambda_2^2\ell_2}[UU'(U'^2 + \lambda^2\Omega^2U^2)]_{\xi_2=\ell_2} \ , \\
\Gamma_3 &= \frac{\omega}{16\lambda_2\lambda^2\ell_2}[-(7\lambda^2+1)UU'\bar{U}' + 4(\lambda^2+1)U'^2\bar{U}]_{\xi_2=\ell_2} \ , \\
\Gamma_4 &= \frac{3\omega}{32\lambda_2\lambda^2\Omega^2\ell_2}\int_{-\ell_2}^{\ell_2}[\beta(U'' + \lambda^2\Omega^2U) - 2\beta'U']\bar{U}'d\xi_2 \ , \\
\Gamma_5 &= \frac{1}{64\lambda_2^2\lambda^2\Omega^2\ell_2}\int_{-\ell_2}^{\ell_2}[\alpha'^2 + 4\Omega^2\alpha^2 + 4\alpha(\beta'U' - \lambda^2\Omega^2\beta U) \\
&\qquad + 16\beta'U'(UU')' + 8\lambda^2\Omega^2(UU')'^2]d\xi_2 \ ,
\end{aligned} \qquad (6.6)$$

and κ and H are given by (5.24) and (5.25).

We now introduce into (6.6) the expressions (3.14), (3.16) and (5.14) for U and $\bar{U}$, together with (3.10), (5.11) and (5.25), and after carrying out the integrations, we obtain, with the notation

$$\bar{M} = M/2\ell_2 , \tag{6.7}$$

the following expressions:

$$\begin{aligned}
\Gamma_1 &= \frac{1}{\kappa}\bar{M}^4\eta^4\lambda_3(7\lambda^2+4\lambda\lambda_3^3+1)(\lambda^2-1)^2[\nu + \frac{\lambda^2-3}{\lambda^2+1}\ \frac{\sinh 2\lambda\eta}{2\lambda\eta}]^2 , \\
\Gamma_2 &= -\frac{\bar{M}^4\eta^3\lambda_3(\lambda^2-1)^2}{2\lambda^2(\lambda^2+1)}\sinh 2\lambda\eta\{\nu(\lambda^2-1)m^2 + 2\lambda^2\cosh 2\lambda\eta \\
&\qquad + (\lambda^2+1)m^2\cosh 2\eta \\
&\qquad - 2\lambda m[(\lambda+1)\cosh(\lambda+1)\eta+\nu(\lambda-1)\cosh(\lambda-1)\eta]\} , \\
\Gamma_3 &= \frac{\bar{M}^4\eta^3\lambda_3}{128\lambda^5(\lambda^2+1)}\{8\lambda(\lambda^2+1)^2\gamma_1\gamma_3+(7\lambda^2+1)(\lambda^2-1)^2\gamma_2\sinh 2\lambda\eta\} , \\
\Gamma_4 &= \frac{3\bar{M}^4\eta^3\lambda_3(\lambda^2-1)}{64\lambda^4}\{\eta\zeta_4 + \sum_{k=1}^{3}[\zeta_4(k,4-k) + \zeta_4(k,k-4)] \\
&\qquad + \sum_{k=1}^{2}[\zeta_4(k-1,3-k) + \zeta_4(k,k-2)]\} , \\
\Gamma_5 &= \frac{\bar{M}^4\eta^3\lambda_3}{32\lambda^3}\{\eta\zeta_5 + \sum_{k=0}^{3}[\zeta_5(k,4-k) + \zeta_5(k+1,k-3)] \\
&\qquad + \sum_{k=1}^{2}[\zeta_5(k-1,3-k) + \zeta_5(k,k-2)]\} ,
\end{aligned} \tag{6.8}$$

where the ζ's are given in the appendix (§ 9), κ is defined by (5.24) and γ_1, γ_2, γ_3 are defined by

$$\gamma_1 = F_1\sinh 2\lambda\eta + F_2\sinh 2\eta$$
$$-m[F_3(\lambda)\sinh(\lambda+1)\eta + \nu F_3(-\lambda)\sinh(\lambda-1)\eta] \ ,$$
$$\gamma_2 = 2\lambda F_1\cosh 2\lambda\eta + 2F_2\cosh 2\eta$$
$$-m[(\lambda+1)F_3(\lambda)\cosh(\lambda+1)\eta + \nu(\lambda-1)F_3(-\lambda)\cosh(\lambda-1)\eta] \ , \qquad (6.9)$$
$$\gamma_3 = \lambda^2\cosh 2\lambda\eta + m^2\cosh 2\eta - \nu(\lambda^2+m^2)$$
$$-2\lambda m[\cosh(\lambda+1)\eta - \nu\cosh(\lambda-1)\eta] \ ,$$

with F_1, F_2 and $F_3(\lambda)$ defined by

$$F_1 = \frac{1}{\Delta}[4\lambda^2 f_1\cosh 2\eta - (\lambda^2+1)f_2\sinh 2\eta] \ ,$$
$$F_2 = \frac{2\lambda}{\Delta}[\lambda f_2\sinh 2\lambda\eta - (\lambda^2+1)f_1\cosh 2\lambda\eta] \ , \qquad (6.10)$$
$$F_3(\lambda) = 24\lambda\frac{(\lambda^2-1)(\lambda+1)}{(3\lambda+1)(\lambda+3)} \ .$$

Here, Δ, f_1 and f_2 are given by (5.17) and m is given by (3.14) if the deformation is flexural and by (3.16) if it is of the barreling type. In either case m^2 is given by (3.19).

Introducing (6.8) into (6.5) we obtain an expression for $\bar{G}$ in the form

$$\bar{G} = 4\ell_1\ell_2\ell_3\bar{M}^4\eta^3\lambda_3(\bar{g}_2 - K\bar{g}_1) \ , \qquad (6.11)$$

where

$$K = \frac{1}{\kappa}(7\lambda^2 + 4\lambda\lambda_3^3 + 1) \ ,$$
$$\bar{g}_1 = \eta(\lambda^2-1)^2\left(\nu + \frac{\lambda^2-3}{\lambda^2+1}\ \frac{\sinh 2\lambda\eta}{2\lambda\eta}\right)^2 \ , \qquad (6.12)$$

$$
\begin{aligned}
2\lambda^2\bar{g}_2 = \frac{(\lambda^2-1)^2}{\lambda^2+1}\ & \sinh 2\lambda\eta\{\nu(\lambda^2-1)m^2+2\lambda^2\cosh 2\lambda\eta+(\lambda^2+1)m^2\cosh 2\eta \\
& -2\lambda m[(\lambda+1)\cosh(\lambda+1)\eta+\nu(\lambda-1)\cosh(\lambda-1)\eta]\} \\
& - \frac{1}{64\lambda^3(\lambda^2+1)}\{8\lambda(\lambda^2+1)^2\gamma_1\gamma_3+(7\lambda^2+1)(\lambda^2-1)^2\gamma_2\sinh 2\lambda\eta\} \\
& - \frac{3(\lambda^2-1)}{32\lambda^2}\{\eta\zeta_4 + \sum_{k=1}^{3}[\zeta_4(k,4-k) + \zeta_4(k,k-4)] \\
& \qquad + \sum_{k=1}^{2}[\zeta_4(k-1,3-k) + \zeta_4(k,k-2)]\} \\
& + \frac{1}{16\lambda}\{\eta\zeta_5 + \sum_{k=0}^{3}[\zeta_5(k,4-k) + \zeta_5(k+1,k-3)] \\
& \qquad + \sum_{k=1}^{2}[\zeta_5(k-1,3-k) + \zeta_5(k,k-2)]\} .
\end{aligned}
$$

In §8 we show the numerical dependence of $\bar{G}$ on η. However, before discussing this we shall consider, in the next section, the asymptotic case when η is small, i.e. when the aspect ratio ℓ_2/ℓ_1 is small, and the bifurcation is of the flexural type.

7. The asymptotic case of small aspect ratio

In principle we could find an expression for $\bar{G}$, for a flexural bifurcation, in the limiting case when $\eta << 1$, directly from the expression (6.5), where the Γ's are given by (6.6), by expanding the Γ's in powers of η and neglecting all but the leading term in the expression for $\bar{G}$ so obtained. However, this leading term would be of degree eleven in η and the labor involved in obtaining expansions of this degree for each of the Γ's would be excessive. We recognize that this difficulty arises from the fact that, for $\eta << 1$, the expression (3.14) for $\hat{U}$ is $O(\eta^2)$ and the expression (5.14) for $\bar{U}$ is $O(\eta^4)$. Accordingly, in this section we return to the equations (3.11) and (3.12) for $\hat{U}$ and obtain a solution, in the form of a power series in η, which is normalized so that $\hat{u}_2(\ell_1,0) = 1$. With this expression for $\hat{U}$, we solve (5.10) and (5.12) to obtain a corresponding expression for $\tilde{U}$, in the form of a power series in η. With these expressions for $\hat{U}$ and $\tilde{U}$, we then obtain from (6.5) and (6.6) an expression for $\bar{G}$, again as a power series in η.

We define the dimensionless thickness coordinate t by

$$t = \xi_2/\ell_2 . \tag{7.1}$$

Then, with the notation

$$\hat{U}(t) = U(\ell_2 t) , \tag{7.2}$$

we obtain from (3.11) and (3.12)

$$\frac{1}{\eta^4}\frac{d^4\hat{U}}{dt^4} - \frac{\lambda^2+1}{\eta^2}\frac{d^2\hat{U}}{dt^2} + \lambda^2\hat{U} = 0 \tag{7.3}$$

and

$$\left.\begin{aligned} &\frac{1}{\eta^2}\frac{d^2\hat{U}}{dt^2} + \lambda^2\hat{U} = 0 , \\ &\frac{1}{\eta^3}\frac{d^3\hat{U}}{dt^3} - (2\lambda^2+1)\frac{1}{\eta}\frac{d\hat{U}}{dt} = 0 , \end{aligned}\right\} \quad \text{for} \quad t = \pm 1. \tag{7.4}$$

The critical value of λ at which a flexural bifurcation can occur is related to η by (3.15). It has been shown [6] that for $\eta << 1$, this relation may be written as

$$\lambda = 1 + a_1\eta^2 + a_2\eta^4 + a_3\eta^6 + O(\eta^8) , \tag{7.5}$$

where

$$a_1 = \frac{2}{3} , \quad a_2 = \frac{16}{45} , \quad \ldots \quad . \tag{7.6}$$

With (7.5), we can obtain a solution of (7.3), with the boundary conditions (7.4), in the form

$$\begin{aligned} \hat{U}(t) = 1 &+ A_{11}\eta^2 t^2 + \eta^4(A_{21}t^2 + A_{22}t^4) \\ &+ \eta^6(A_{31}t^2 + A_{32}t^4 + A_{33}t^6) + O(\eta^8) , \end{aligned} \tag{7.7}$$

where

$$\begin{aligned} &A_{11} = -\frac{1}{2} , \quad A_{21} = \frac{1}{3} , \quad A_{22} = -\frac{1}{8} , \\ &A_{31} = \frac{4}{45} , \quad A_{32} = -\frac{1}{18} , \quad A_{33} = -\frac{1}{144} . \end{aligned} \tag{7.8}$$

We note, from (3.8), that we have normalized the solution (7.7), so that the displacement $\hat{u}_2(\ell_1,0) = \pm 1$, the $+(-)$ sign being applicable if $\frac{1}{2}n$, or $\frac{1}{2}(n-1)$, is even (odd).

With (7.1) and the notation

$$\tilde{U}(t) = \bar{U}(\ell_2 t) \ , \ \hat{\alpha} = \frac{\alpha}{\Omega^2} = \frac{1}{\eta^2}\left\{\hat{U}\ \frac{d^2\hat{U}}{dt^2} - \left(\frac{d\hat{U}}{dt}\right)^2\right\} , \tag{7.9}$$

we can rewrite (5.10) and (5.12) as

$$\frac{1}{\eta^4}\ \frac{d^4\tilde{U}}{dt^4} - \frac{4(\lambda^2+1)}{\eta^2}\ \frac{d^2\tilde{U}}{dt^2} + 16\lambda^2\tilde{U} = -\ \frac{3\omega\Omega}{2\lambda_2\eta}\ (\lambda^2-1)\ \frac{d\hat{\alpha}}{dt} \tag{7.10}$$

and

$$\left.\begin{aligned} &\frac{1}{\eta^2}\ \frac{d^2\tilde{U}}{dt^2} + 4\lambda^2\tilde{U} = -\ \frac{\omega\Omega}{2\lambda_2\eta}\ (7\lambda^2+1)\hat{U}\ \frac{d\hat{U}}{dt} , \\ &\frac{1}{\eta^3}\ \frac{d^3\tilde{U}}{dt^3} - 4(2\lambda^2+1)\ \frac{1}{\eta}\ \frac{d\tilde{U}}{dt} = \frac{2\omega\Omega}{\lambda_2\eta^2}\ (2\lambda^2-1)\left(\frac{d\hat{U}}{dt}\right)^2 , \end{aligned}\right\} \text{ for } t = \pm 1 . \tag{7.11}$$

With (7.5)-(7.9), equations (7.10) and (7.11) become

$$\begin{aligned} \frac{1}{\eta^4}\ \frac{d^4\tilde{U}}{dt^4} - \frac{4(\lambda^2+1)}{\eta^2}\ \frac{d^2\tilde{U}}{dt^2} + 16\lambda^2\tilde{U} = 8\omega\Omega\lambda_3^{\frac{1}{2}}\eta^2\Big\{&\eta t \\ &+ \eta^3(\tfrac{8}{15}t+\tfrac{1}{3}t^3) + O(\eta^5)\Big\} \end{aligned} \tag{7.12}$$

and

$$\left.\begin{aligned} &\frac{1}{\eta^2}\ \frac{d^2\tilde{U}}{dt^2} + 4\lambda^2\tilde{U} = \omega\Omega\lambda_3^{\frac{1}{2}}\eta\left\{4 + \tfrac{2}{3}\eta^2 + \tfrac{26}{15}\eta^4 + O(\eta^6)\right\} , \\ &\frac{1}{\eta^3}\ \frac{d^3\tilde{U}}{dt^3} - 4(2\lambda^2+1)\frac{1}{\eta}\ \frac{d\tilde{U}}{dt} = \omega\Omega\lambda_3^{\frac{1}{2}}\eta^2\left\{2 + 4\eta^2 + O(\eta^4)\right\} \\ &\qquad\qquad\qquad\qquad\qquad\qquad\text{for } t = \pm 1 . \end{aligned}\right. \tag{7.13}$$

It follows from (7.12) and (7.13), with (7.5) and (7.6), that $\tilde{U}$ may be expressed in the form

$$\tilde{U} = \omega\Omega\lambda_3^{1/2}\{B_{11}\eta t + \eta^3(B_{21}t + B_{22}t^3) + \eta^5(B_{31}t + B_{32}t^3 + B_{33}t^5)$$
$$+ \eta^7(B_{41}t + \ldots + B_{44}t^7) + O(\eta^9)\}\ , \tag{7.14}$$

where

$$B_{11} = \frac{1}{4}\ ,\quad B_{21} = -\frac{7}{6}\ ,\quad B_{22} = \frac{1}{2}\ ,\quad B_{31} = \frac{23}{72}\ ,$$
$$B_{32} = -\frac{2}{9}\ ,\quad B_{33} = \frac{1}{6}\ ,\quad B_{43} = \frac{2}{9}\ ,\quad B_{44} = \frac{1}{45}\ . \tag{7.15}$$

We shall not require the values of B_{41}, B_{42} .

With (7.1), we now rewrite (6.6) as

$$\Gamma_1 = \frac{\lambda_3\Omega^4}{16\lambda^4\kappa}(7\lambda^2 + 4\lambda\lambda_3^3 + 1)\hat{H}^2\ ,$$

$$\Gamma_2 = \frac{\lambda_3\Omega^4}{4\lambda\eta^4}\left\{\hat{U}\frac{d\hat{U}}{dt}\left[\left(\frac{d\hat{U}}{dt}\right)^2 + \lambda^2\eta^2\hat{U}^2\right]\right\}_{t=1}\ ,$$

$$\Gamma_3 = \frac{\omega\lambda_3^{1/2}\Omega^3}{16\lambda^{5/2}\eta^3}\left\{4(\lambda^2+1)\left(\frac{d\hat{U}}{dt}\right)^2\tilde{U} - (7\lambda^2+1)\hat{U}\frac{d\hat{U}}{dt}\frac{d\tilde{U}}{dt}\right\}_{t=1}\ ,$$

$$\Gamma_4 = \frac{3\omega\lambda_3^{1/2}\Omega^3}{32\lambda^{5/2}\eta^3}\int_{-1}^{1}\left\{\hat{\beta}\left(\frac{d^2\hat{U}}{dt^2} + \lambda^2\eta^2\hat{U}\right) - 2\frac{d\hat{\beta}}{dt}\frac{d\hat{U}}{dt}\right\}\frac{d\tilde{U}}{dt}\,dt\ ,$$

$$\Gamma_5 = \frac{\lambda_3\Omega^4}{64\lambda^3\eta^4}\int_{-1}^{1}\left\{\eta^2\left(\frac{d\hat{\alpha}}{dt}\right)^2 + 4\eta^4\hat{\alpha}^2 + 4\eta^2\hat{\alpha}\left(\frac{d\hat{\beta}}{dt}\frac{d\hat{U}}{dt} - \lambda^2\eta^2\hat{\beta}\hat{U}\right)\right.$$
$$\left. + 16\frac{d\hat{\beta}}{dt}\frac{d\hat{U}}{dt}\frac{d}{dt}\left(\hat{U}\frac{d\hat{U}}{dt}\right) + 8\lambda^2\left[\frac{d}{dt}\left(\hat{U}\frac{d\hat{U}}{dt}\right)\right]^2\right\}dt\ , \tag{7.16}$$

where $\hat{\alpha}$ is given by (7.9) and $\hat{\beta}$ and $\hat{H}$ are defined by

$$\hat{\beta} = \frac{1}{\eta^2}\frac{d^2\hat{U}}{dt^2} - \hat{U}\ ,$$
$$\hat{H} = \int_{-1}^{1} \frac{1}{\eta^4}\left\{\left(\frac{d^2\hat{U}}{dt^2}\right)^2 + \eta^2\left(\frac{d\hat{U}}{dt}\right)^2\right\}dt\ . \tag{7.17}$$

We now substitute from (7.5), (7.7) and (7.14) in (7.16) and, after carrying out the indicated integrations, we obtain

$$\Gamma_1 = \lambda_3\Omega^4\left[\frac{1}{4} - \frac{3}{4}\eta^2 + \left(\frac{91}{90} + \frac{1}{36(2+\lambda_3^3)}\right)\eta^4 + O(\eta^6)\right],$$
$$\Gamma_2 = \lambda_3\Omega^4\left[-\frac{1}{4} + \frac{8}{45}\eta^4 + O(\eta^6)\right],$$
$$\Gamma_3 = \lambda_3\Omega^4\left[\frac{1}{8} + \frac{7}{48}\eta^2 - \frac{25}{72}\eta^4 + O(\eta^6)\right], \tag{7.18}$$
$$\Gamma_4 = \lambda_3\Omega^4\left[-\frac{3}{16}\eta^2 + \frac{17}{24}\eta^4 + O(\eta^6)\right],$$
$$\Gamma_5 = \lambda_3\Omega^4\left[\frac{1}{8} - \frac{3}{4}\eta^2 + \frac{199}{180}\eta^4 + O(\eta^6)\right].$$

With (7.18), we obtain from (6.5) and (4.10)

$$G = a^4\bar{G} = \frac{a^4\ell_1\ell_3\lambda_3}{6\ell_2^3}\eta^6\left[G^* + O(\eta^4)\right],$$
$$G^* = 1 - \frac{2}{3}\left(16 + \frac{1}{2+\lambda_3^3}\right)\eta^2\ . \tag{7.19}$$

We can compare this result with a result which was derived by Euler on the basis of the elastica theory. In order to do this, we first calculate the energy associated with the flexural deformation $a\hat{u}_1$, $a\hat{u}_2$ when the applied load is zero. Accordingly, in (2.12) we take $\lambda_1 = \lambda_2 = 1$ and substitute u_1, u_2, E =

$a\hat{u}_1$, $a\hat{u}_2$, 0, where $\hat{u}_1$, $\hat{u}_2$ are the values of u_1, u_2 given in (3.8) with $U = \hat{U}$ given by (7.7). Denoting the resulting value of G by $\hat{G}$, we obtain

$$\hat{G} = \frac{8a^2 \ell_1 \ell_3}{3\ell_2} \eta^4 \left[1 + O(\eta^2) \right] . \tag{7.20}$$

Then from (7.19) and (7.20) we obtain, with $\lambda_3 = 1$,

$$\frac{G}{\hat{G}} = \frac{a^2 \eta^2}{16 \ell_2^2} \left[1 + O(\eta^2) \right] . \tag{7.21}$$

For the flexural mode of lowest order, we have $\eta = \pi \ell_2 / 2\ell_1$, and (7.21) may be rewritten as

$$\frac{G}{\hat{G}} = \frac{\pi^2 a^2}{64 \ell_1^2} \left[1 + O\left(\frac{\ell_2}{\ell_1} \right)^2 \right] , \tag{7.22}$$

in agreement with the result of Euler.

8. Numerical results

The critical value of λ at which a bifurcation occurs is related to η by (3.15) or (3.17), accordingly as the bifurcation is of the flexural or barreling type. These relations are plotted as Curves I and II in Fig.1. Similar curves have been shown previously by a number of authors (see, for example, [2]). The approximate relation (cf.(7.5) with (7.6))

$$\lambda = 1 + \frac{2}{3}\eta^2 + \frac{16}{45}\eta^4 , \tag{8.1}$$

valid in the flexural case for small η, is plotted as Curve III.

The asymptotic expression (7.19) for $\bar{G}$, valid for flexural deformations, has been calculated on the basis of an expression for $\hat{\underset{\sim}{u}}$, normalized so that its component $\hat{u}_2(\ell_1,0) = \pm 1$, accordingly as $\frac{1}{2}n$, or $\frac{1}{2}(n-1)$, is even or odd. In order to compare with this expression for $\bar{G}$, the exact expression given by equations (6.11) and (6.12), we have to normalize, in the same manner, the expression for $\hat{\underset{\sim}{u}}$ on which it is based. From (3.8) and $(3.14)_1$, we have

$$\hat{u}_2(\ell_1,0) = \pm M(1-m) , \tag{8.2}$$

where m is given by $(3.14)_2$ and the $+(-)$ sign is taken accordingly as $\frac{1}{2}n$, or $\frac{1}{2}(n-1)$, is even (odd). Accordingly, in order to normalize our results in the desired manner we must take

$$M = \frac{1}{1-m} \quad \text{and} \quad \bar{M} = \frac{1}{2\ell_2(1-m)} . \tag{8.3}$$

With this expression for $\overline{M}$, we can rewrite equation (6.11) as (cf.$(7.19)_1$)

$$\overline{G} = \frac{\ell_1 \ell_3 \lambda_3}{6\ell_2^3} \eta^6 G^* , \tag{8.4}$$

where G^* is given by

$$G^* = \frac{3}{2\eta^3 (1-m)^4} (\overline{g}_2 - K\overline{g}_1) , \tag{8.5}$$

and m is given by $(3.14)_2$, while $\overline{g}_1$, $\overline{g}_2$ and K are given by (6.12) with $\nu = 1$.

The relation between G^* and η provided by (8.5), with $\lambda_3 = 1$, is plotted as Curve I of Fig.2 for $\eta < 0.6$. Curve II shows the relation between G^* and η provided by the asymptotic formula $(7.19)_2$, with $\lambda_3 = 1$, and we note that the agreement is good at the lower values of η. In Fig.3 the relation between G^* and η, provided by the exact formula (8.5), with $\lambda_3 = 1$, is plotted for a wider range of values of η. Calculations for larger values of η than those covered in Fig.3 indicate that the G^* vs. η curve continues smoothly, with G^* tending to $-\infty$ as $\eta \to \infty$ (i.e., $\lambda \to 3.383$). We note (see §5) that our calculations have excluded the case when $\lambda = 3$ (i.e., $\eta \simeq 1.63$). It was not considered worthwhile to investigate this case separately, since our computations indicate that the G^* vs. η curve passes smoothly through this point.

From Fig.2 or 3, we see that G^* is positive for values of η below about 0.32. For the lowest flexural mode this corresponds, from (3.9) and (3.13) with $n = 1$, to an aspect ratio $\ell_2/\ell_1 \simeq 0.20$. For larger values of η, G^* is negative. Accordingly, state I, the homogeneous state corresponding to the critical value of λ at which a flexural bifurcation occurs, is unstable if $\eta > 0.32$, and stable if $\eta < 0.32$ and this stability is not preempted by the appearance of a mode of lower order.

We note that for a specified value of η, the critical value of λ, given by (3.15), is independent of λ_3. It follows from $(3.14)_2$, (5.17) and (5.18) that Δ, f_1 and f_2 are independent of λ_3. Then, from (6.9), (6.10) and equations (9.1)-(9.4) in the Appendix, it follows that γ_1, γ_2, γ_3 and the ζ's in $(6.12)_3$ are independent of λ_3. Consequently, $\bar{g}_1$ and $\bar{g}_2$ are independent of λ_3. Also, for a fixed value of λ, the quantity K, defined by $(6.12)_1$, decreases monotonically with increase of λ_3. Hence, from (8.5), G^* changes monotonically with λ_3. However, the dependence of K, and hence of G^*, on λ_3 is very slight.

For barreling deformations, the exact expression for $\bar{G}$ is given by (6.11) and (6.12), with $\nu = -1$. We cannot normalize the displacement $\hat{\underset{\sim}{u}}$ on which it is based in the same way as we did for flexural deformations, since $\hat{u}_2(\ell_1,0) = 0$ for barreling deformations. Instead, we now normalize $\hat{\underset{\sim}{u}}$ so that $\hat{u}_2(\ell_1,\ell_2) = \pm 1$ accordingly as $\frac{1}{2}n$, or $\frac{1}{2}(n-1)$, is even or odd by taking

$$\bar{M} = \frac{M}{2\ell_2} = \frac{1}{2\ell_2(\sinh\lambda\eta - m\sinh\eta)}, \tag{8.6}$$

where m is now given by the expression $(3.16)_2$, appropriate to barreling deformations. With this expression for $\overline{M}$ we can now rewrite (6.11) as

$$\overline{G} = \frac{\ell_1 \ell_3 \lambda_3}{4\ell_2^3} G^* , \tag{8.7}$$

where

$$G^* = \frac{\eta^3(\overline{g}_2 - K\overline{g}_1)}{(\sinh\lambda\eta - m\sinh\eta)^4} . \tag{8.8}$$

G^* is plotted against η in Fig.4 for the case when $\lambda_3 = 1$. As in the case of flexural deformations the dependence of G^* on λ_3 is very slight. We note that G^* , and hence $\overline{G}$, is positive for all η . Accordingly, the homogeneous state corresponding to a critical value of λ at which a barreling bifurcation occurs is stable.

We have included the discussion of barreling bifurcations for completeness. However, it is well to realize that, for a specified aspect ratio ℓ_2/ℓ_1 , as the load is increased values of λ corresponding to flexural bifurcations of all orders are reached before any barreling bifurcation is attained. Even if these could be inhibited, the barreling bifurcation of highest (theoretically infinite) order is reached first, corresponding to wrinkling of the free surfaces.

9. Appendix

The quantities ζ_4 and ζ_5 introduced in §6 are defined by

$$\zeta_4 = \frac{1}{2} m^2(\lambda^2-1)\{(\lambda-1)F_3(\lambda) + (\lambda+1)F_3(-\lambda)\} ,$$

$$\begin{aligned}\zeta_5 = 4\{&16[\lambda^2(\lambda^2-1)m^2 + \lambda^2(\lambda^4+m^4) + (\lambda^2+m^2)^2] \\ &-2\lambda m^2(\lambda^2-1)[(\lambda+1)^3 + (\lambda-1)^3] \\ &+ 2[(\lambda-1)^4 + (\lambda+1)^4](2\lambda^2+1)m^2 -(\lambda^2-1)^2(\lambda^2+1)m^2\} .\end{aligned} \tag{9.1}$$

The quantities $\zeta_4(j,k)$ and $\zeta_5(j,k)$ introduced in §6 are defined by

$$\begin{aligned}\zeta_4(j,k) &= \frac{\overline{\zeta}_4(j,k)}{j\lambda+k} \sinh\{(j\lambda+k)\eta\} \\ \zeta_5(j,k) &= \frac{\overline{\zeta}_5(j,k)}{j\lambda+k} \sinh\{(j\lambda+k)\eta\}\end{aligned} \tag{9.2}$$

where j and k are integers and

$$\begin{aligned}
&\overline{\zeta}_4(3,1) = -\lambda(\lambda-1)^2 mF_1 , \quad \overline{\zeta}_4(3,-1) = -\nu\lambda(\lambda+1)^2 mF_1 , \\
&\overline{\zeta}_4(1,3) = -(\lambda-1)^2 mF_2 , \quad \overline{\zeta}_4(1,-3) = -\nu(\lambda+1)^2 mF_2 , \\
&\overline{\zeta}_4(2,2) = \frac{1}{2} (\lambda+1)(\lambda-1)^2 m^2 F_3(\lambda) , \\
&\overline{\zeta}_4(2,-2) = \frac{1}{2} (\lambda-1)(\lambda+1)^2 m^2 F_3(-\lambda) , \\
&\overline{\zeta}_4(2,0) = \frac{1}{2}\nu\{16\lambda^3 F_1 + m^2[(\lambda+1)^3 F_3(\lambda) + (\lambda-1)^3 F_3(-\lambda)]\} , \\
&\overline{\zeta}_4(0,2) = \frac{1}{2}\nu\{16\lambda^2 F_2 + m^2[(\lambda+1)^3 F_3(\lambda) + (\lambda-1)^3 F_3(-\lambda)]\} , \\
&\overline{\zeta}_4(1,1) = -\nu(\lambda+1)m\{4\lambda^2 F_3(\lambda) + (\lambda+1)(\lambda F_1+F_2)\} , \\
&\overline{\zeta}_4(1,-1) = -(\lambda-1)m\{4\lambda^2 F_3(-\lambda) + (\lambda-1)(\lambda F_1+F_2)\} ,
\end{aligned} \tag{9.3}$$

and

$$\bar{\zeta}_5(4,0) = 64\lambda^4(2\lambda^2-1) \;, \quad \bar{\zeta}_5(0,4) = 64\lambda^2 m^4 \;,$$

$$\bar{\zeta}_5(3,1) = -32\lambda^2(\lambda+1)m\{(\lambda+1)(3\lambda^2-1) + 2\lambda(\lambda-1)\} \;,$$

$$\bar{\zeta}_5(3,-1) = -32\nu\lambda^2(\lambda-1)m\{(\lambda-1)(3\lambda^2-1) - 2\lambda(\lambda+1)\} \;,$$

$$\bar{\zeta}_5(1,3) = -64\lambda(\lambda+1)m^3(\lambda^2+2\lambda-1) \;,$$

$$\bar{\zeta}_5(1,-3) = -64\nu\lambda(\lambda-1)m^3(\lambda^2-2\lambda-1) \;,$$

$$\begin{aligned}\bar{\zeta}_5(2,2) = 2m^2\{&(\lambda-1)^4[4-(\lambda+1)(3\lambda-1)] \\ &+ 8\lambda(\lambda+1)^3(\lambda^2+3\lambda-2) + 32\lambda^2(3\lambda^2-1)\} \;,\end{aligned}$$

$$\begin{aligned}\bar{\zeta}_5(2,-2) = 2m^2\{&(\lambda+1)^4[4-(\lambda-1)(3\lambda+1)] \\ &+ 8\lambda(\lambda-1)^3(\lambda^2-3\lambda-2) + 32\lambda^2(3\lambda^2-1)\} \;,\end{aligned} \tag{9.4}$$

$$\bar{\zeta}_5(2,0) = 4\nu(\lambda^2-1)\{m^2(\lambda^2-1)(5\lambda^2+3) - 32\lambda^2(\lambda^2+m^2)\} \;,$$

$$\bar{\zeta}_5(0,2) = 4\nu(\lambda^2-1)m^2\{(\lambda^2-1)(3\lambda^2+5) - 64\lambda^2\} \;,$$

$$\begin{aligned}\bar{\zeta}_5(1,1) = -32\nu(\lambda-1)m\{&(\lambda^2+m^2)(\lambda-1)(\lambda^2-\lambda+2) \\ &- 2\lambda(\lambda+1)[m^2+\lambda(\lambda^2+3\lambda+1)]\} \;,\end{aligned}$$

$$\begin{aligned}\bar{\zeta}_5(1,-1) = -32(\lambda+1)m\{&(\lambda^2+m^2)(\lambda+1)(\lambda^2+\lambda+2) \\ &+ 2\lambda(\lambda-1)[m^2-\lambda(\lambda^2-3\lambda+1)]\} \;.\end{aligned}$$

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Figure Captions

Fig. 1 Relation between η and critical values of λ . Curve I: flexure; Curve II: barreling; Curve III: asymptotic expression, correct to $O(\eta^4)$.

Fig. 2 Relation between G^* and η for flexure, with $\lambda_3 = 1$. Curve I: exact expression; Curve II: asymptotic expression, correct to $O(\eta^2)$.

Fig. 3 Relation between G^* and η for flexure, with $\lambda_3 = 1$.

Fig. 4 Relation between G^* and η for barreling, with $\lambda_3 = 1$.

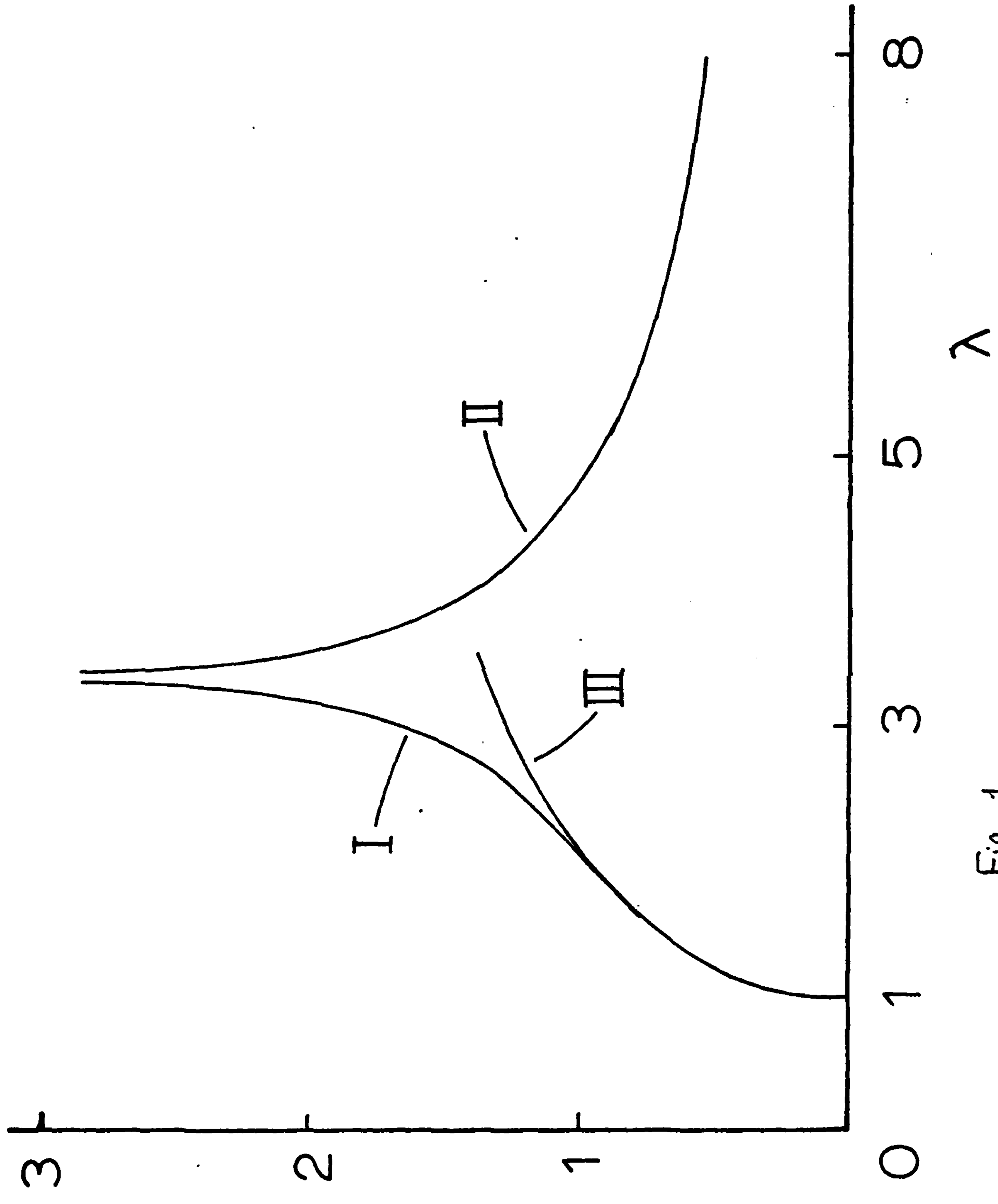

Fig. 1

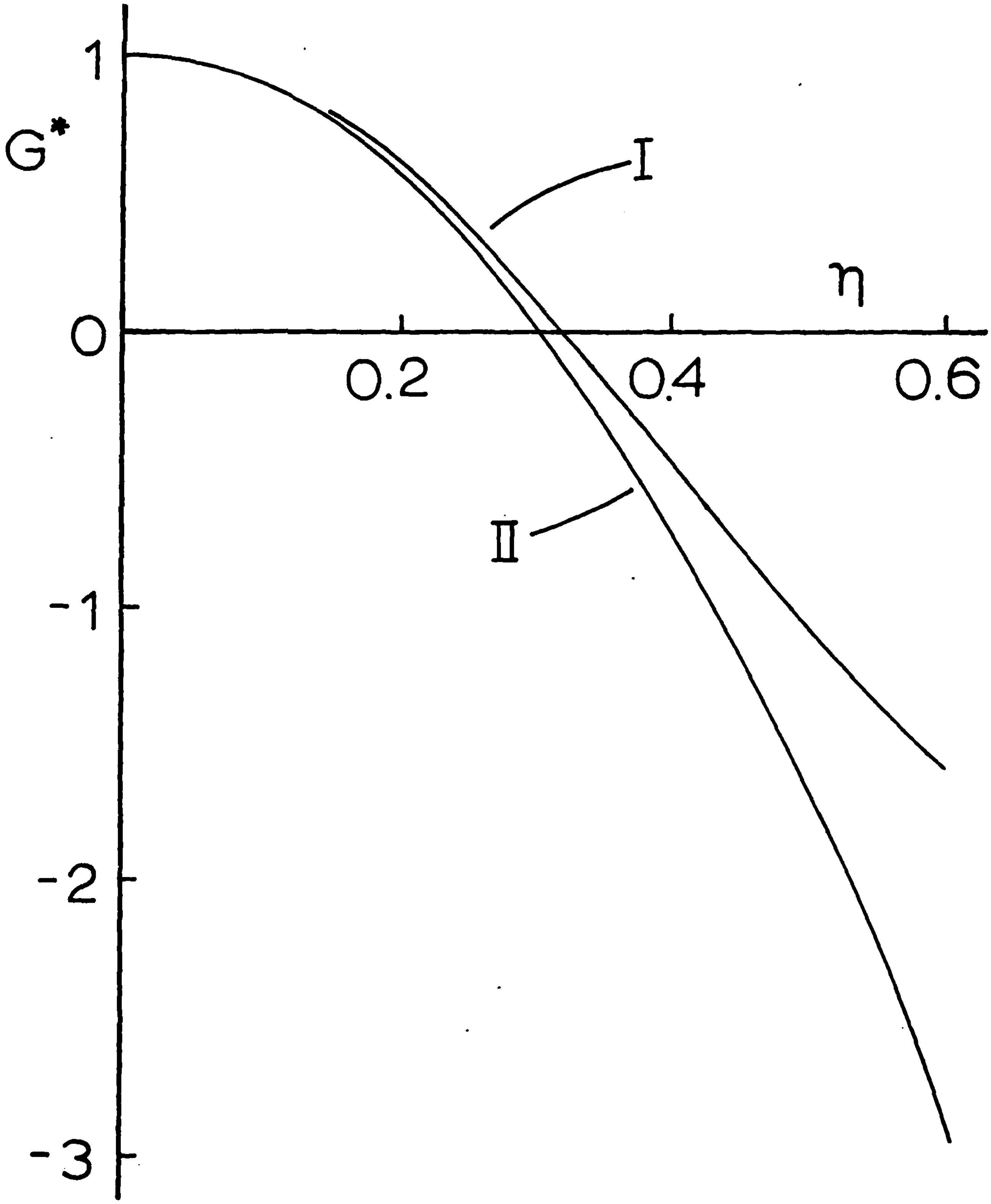

Fig. 2

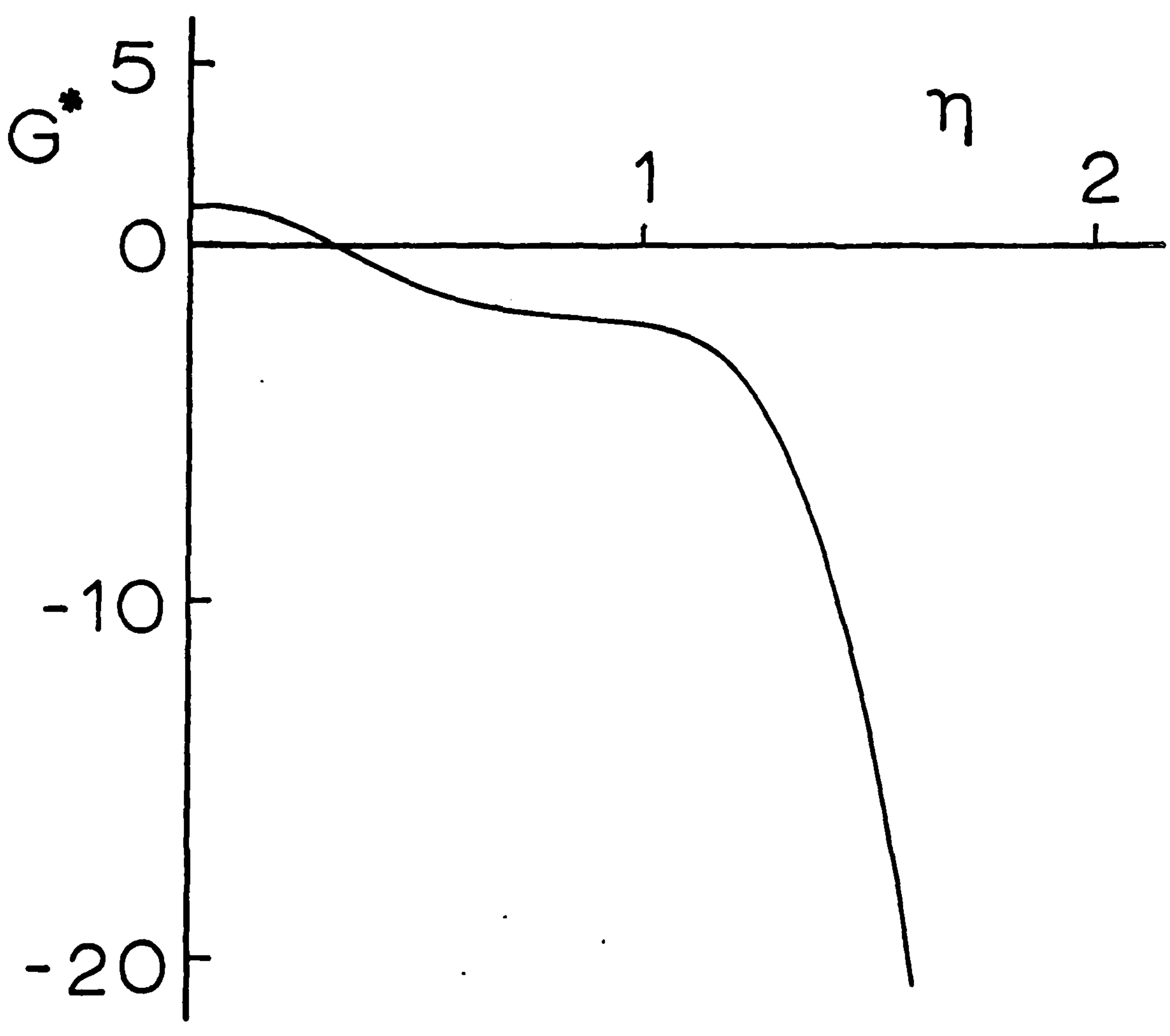

Fig. 3

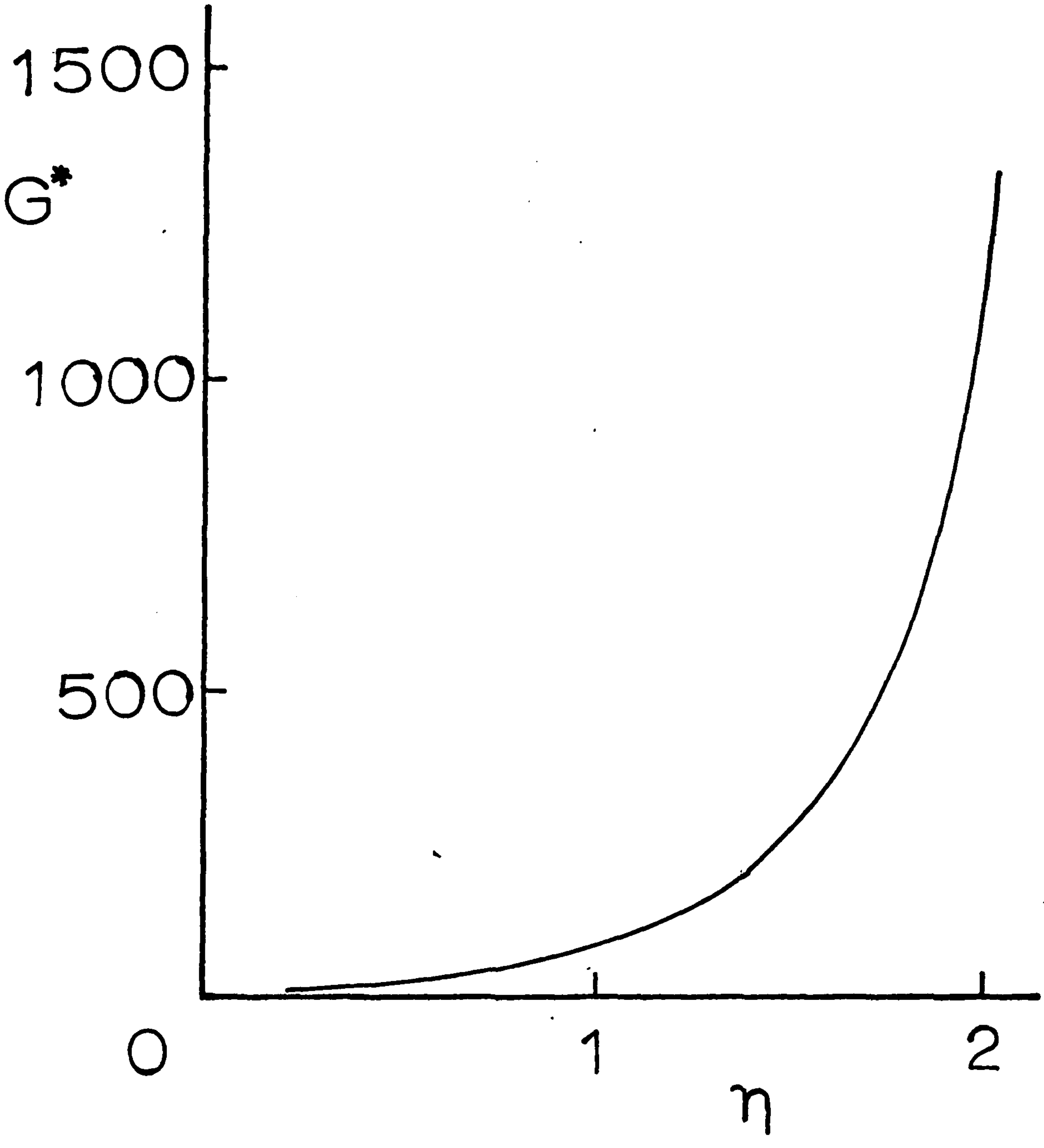

Fig. 4

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

When a rectangular plate of incompressible neo-Hookean elastic material is subjected to a thrust, bifurcations of the flexural or barreling types become possible at certain critical values of the compression ratio. The states of pure homogeneous deformation corresponding to these critical compression ratios are states of neutral equilibrium. Their stability is investigated on the basis of an energy criterion, without restriction on the thickness of the plate.

The critical state corresponding to the lowest order flexural mode is

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found to be stable (unstable) if the aspect ratio (thickness/length) is less (greater) than about 0.2. Agreement with the classical Euler theory is established in the limiting case of small aspect ratio.